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ROUGH I-STATISTICAL CONVERGENCE OF ORDER α' IN NEUTROSOPHIC NORMED SPACES: CHARACTERIZATIONS AND APPLICATIONS

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ABSTRACT. This paper investigates rough I-statistical convergence of order α' in neutrosophic normed spaces. We establish the existence of a minimal roughness parameter that guarantees the nonemptiness of the I-statistical limit set and demonstrate that this mode of convergence remains stable under small perturbations. A uniqueness result for the common limit point is derived whenever the intersection of all corresponding limit sets is nonempty. Finally, the concept is characterized in terms of subsequences determined by filter sets and by decomposing sequences into statistically convergent and statistically bounded parts.

Furthermore, the proposed theory is applied to neutrosophic dynamical systems, fixed-point iterations, and sensor-network data fusion. The obtained results establish stability under sparse disturbances, preservation of convergence for perturbed Picard processes, and robust estimation in the presence of sporadic faulty observations. These applications demonstrate the effectiveness of rough I-statistical convergence of order α' in modeling uncertainty, robustness, and long-term behavior in neutrosophic environments.

1. INTRODUCTION

The theory of statistical convergence has become an important area of research due to its wide applicability in summability theory, functional analysis, approximation theory, and dynamical systems. The concept provides a natural generalization of ordinary convergence by allowing exceptional terms to occur on sets of natural density zero. The notion was later extended in various directions through the introduction of ideal convergence by Kostyrko et al. [2] and ideal statistical convergence by Das et al. [3], which offer a flexible framework for studying convergence with respect to admissible ideals.

Another significant generalization is rough convergence, introduced by Phu [4, 5], where convergence is investigated within a prescribed tolerance radius rather than at a single limit point. This concept has attracted considerable attention because it is particularly suitable for analyzing data contaminated by measurement errors, perturbations, and uncertainties. Subsequently, rough statistical convergence was introduced by Aytar [1] and further developed by Maity [6], while rough ideal convergence and its variants were studied by Pal et al. [7], Dündar and Çakan [21], Dündar [20], Dündar and Ulusu [22, 23], Bulut and Or [8], and Ahmad et al. [9, 10]. Related investigations have also been carried out in probabilistic normed spaces, intuitionistic fuzzy normed spaces, and 2-normed spaces [15, 16, 17, 18, 19].

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Parallel to these developments, the theory of fuzzy, intuitionistic fuzzy, and neutrosophic structures has emerged as a powerful mathematical tool for modeling uncertainty and indeterminacy. Important contributions to intuitionistic fuzzy metric and normed spaces were made by Park [26], Saadati and Park [27], Lael and Nourouzi [24], Karakuş et al. [25], Mursaleen and Lohani [28], Mursaleen and Mohiuddine [29], and Kumar and Mursaleen [30]. The introduction of neutrosophic theory by Smarandache [11] significantly broadened the scope of uncertainty modeling by incorporating truth, indeterminacy, and falsity components simultaneously. Recently, neutrosophic normed spaces and their convergence structures have been investigated by Kirişci and Şimşek [12], Khan et al. [13, 14, 31, 32, 33], among others and Recent developments in neutrosophic normed spaces have focused on generalized convergence, summability, and topological structures. Ahmad and Mursaleen [34] introduced deferred statistical summability in neutrosophic (n)-normed linear spaces, while Ahmad and Mursaleen [35] studied completeness and Cartesian products in neutrosophic rectangular (n)-normed spaces. These works provide important foundations for further investigations in neutrosophic functional analysis.

Despite the extensive literature on rough convergence, statistical convergence, ideal convergence, and neutrosophic normed spaces, the study of rough $\mathtt{I}$-statistical convergence of order α' in neutrosophic normed spaces has not yet been systematically developed. The presence of the order parameter α' allows a finer control over the asymptotic behavior of exceptional sets, while the roughness parameter provides a natural mechanism for handling uncertainty and approximation. Consequently, a unified theory combining rough convergence, ideal statistical convergence, and neutrosophic norms is both mathematically meaningful and practically relevant.

Motivated by these observations, in this paper we introduce the notions of rough statistical convergence and rough $\mathtt{I}$-statistical convergence of order α' in neutrosophic normed spaces and investigate their fundamental properties. We establish the existence of a minimal roughness parameter ensuring the nonemptiness of the corresponding rough $\mathtt{I}$-statistical limit set, prove stability under perturbations, and derive a uniqueness theorem for common rough $\mathtt{I}$-statistical limit points. Furthermore, we obtain characterization theorems via filter-generated subsequences and decomposition methods involving $\mathtt{I}$-statistically convergent and $\mathtt{I}$-statistically bounded sequences.

In addition, several applications of the proposed theory are presented. In particular, we investigate rough $\mathtt{I}$-statistical stability in neutrosophic dynamical systems, convergence of perturbed Picard iteration processes in fixed-point theory, and robust data-fusion models in neutrosophic sensor networks. These applications demonstrate that rough $\mathtt{I}$-statistical convergence of order α' provides an effective framework for studying stability, robustness, uncertainty, and long-term behavior in complex neutrosophic environments.

2. Notation and Basic Definitions

This section recalls the necessary notation definitions and basic concepts concerning rough ideal statistical convergence and neutrosophic normed linear spaces.

Definition 2.1. [2] *Let $\mathcal{Y} \neq \emptyset$. A family $\mathtt{I} \subset P(\mathcal{Y})$ of subsets of $\mathcal{Y}$ is called an ideal in $\mathcal{Y}$ provided,*
(i) $\emptyset \in \mathtt{I}$,
(ii) $R, S \in \mathtt{I} \Longrightarrow R \cup S \in \mathtt{I}$,
(iii) For each $R \in \mathtt{I}, S \subset R \Longrightarrow S \in \mathtt{I}$.

Here, $P(\mathcal{Y})$ denotes the power set of $\mathcal{Y}$, i.e., the set of all its subsets. An ideal $\mathtt{I}$, which is a proper (non-trivial) subset of $P(\mathcal{Y})$—that is, $\mathtt{I} \neq P(\mathcal{Y})$—is referred to as admissible if it includes every singleton subset of $\mathcal{Y}$.

Basic Sets and Parameters	
$\mathcal{N}$	Set of all natural numbers.
$\alpha' \in (0,1]$	Order of statistical and ideal statistical convergence.
$j' \geq 0$	Roughness parameter controlling the allowable deviation in convergence.
$\lambda^* \in (0,1)$	Neutrosophic precision parameter.
$\varepsilon > 0$	Arbitrarily small positive real number.
Ideals and Filters	
$\mathtt{I}$	Non-trivial admissible ideal on $\mathcal{N}$.
$\mathcal{F}(\mathtt{I})$	Filter associated with the ideal $\mathtt{I}$.
$\delta^*_{\mathtt{I}}(A)$	$\mathtt{I}$-asymptotic density of order α' for $A \subseteq \mathcal{N}$.
Spaces and Structures	
$\mathcal{Y}$	Linear space over a field F.
$(\mathcal{Y}, \|\cdot\|)$	Normed linear space.
$(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$	Neutrosophic normed space (NNS).
Neutrosophic Norm Functions	
$\Upsilon(y, \lambda^*)$	Degree of membership of y.
$\Omega(y, \lambda^*)$	Degree of non-membership of y.
$\Gamma(y, \lambda^*)$	Degree of indeterminacy of y.
Sequences and Convergence	
$y = \{y'_m\}$	A sequence in $\mathcal{Y}$.
$y'_m \xrightarrow{(\Upsilon,\Omega,\Gamma)} \xi$	Convergence in a neutrosophic normed space.
$y'_m \xrightarrow{j'(\Upsilon,\Omega,\Gamma)} \xi$	Rough convergence in a neutrosophic normed space.
$y'_m \xrightarrow{j'-st(\alpha')} \xi$	Rough statistical convergence of order α'.
$y'_m \xrightarrow{j'-\mathtt{I}-st(\alpha')} \xi$	Rough $\mathtt{I}$-statistical convergence of order α'.
$\mathrm{LIM}_y^{j'}$	Set of all rough limit points of the sequence y.
$\mathtt{I}$-st-$\mathrm{LIM}_{j'}^{\alpha'}\, y$	Set of rough $\mathtt{I}$-statistical limit points of order α'.

Throughout this paper, we assume that $\mathtt{I}$ represents a non-trivial admissible ideal defined on the set of natural numbers.

Definition 2.2. [2] *Let $\mathcal{Y} \neq \emptyset$. A non-empty class $\mathcal{F} \subset P(\mathcal{Y})$ is called filter on $\mathcal{Y}$ provided,*
(i) $\emptyset \notin \mathcal{F}$,
(ii) $S, T \in \mathcal{F} \Longrightarrow S \cap T \in \mathcal{F}$,
(iii) For each $S \in \mathcal{F}, S \subset T \Longrightarrow T \in \mathcal{F}$.

Every ideal $\mathtt{I}$ *is associated with a filter* $\mathcal{F}(\mathtt{I})$ *defined as follows:*

$$\mathcal{F}(\mathtt{I}) = \{\mathbb{M} \subseteq \mathcal{Y} : \mathbb{M}^c \in \mathtt{I}\} \text{ where } \mathbb{M}^c = \mathcal{Y} - \mathbb{M}.$$

Definition 2.3. [2] *A sequence $y = \{y'_m\}$ in $\mathcal{Y}$ is called ideal convergent ($\mathtt{I}-$convergent) to η if for every $\varepsilon > 0$*

$$\mathring{A}(\varepsilon) = \{n \in \mathcal{N} : |y'_m - \eta| \geq \varepsilon\} \in \mathtt{I}.$$

Here, η is known as the $\mathtt{I}$-limit of the sequence $y = \{y'_m\}$.
Now, we recall rough convergence as follows:

Definition 2.4. [4] *Let $(\mathcal{Y}, \|.\|)$ be a normed space. A sequence $y = y'_m$ in $\mathcal{Y}$ is said to be roughly convergent (or j'-convergent) to a point $\xi_1 \in \mathcal{Y}$ with roughness level $j' \geq 0$ if, for every $\varepsilon > 0$, there exists an index $m_0 \in \mathcal{N}$ such that the inequality $|y'_m - \xi_1| < j' + \varepsilon$ holds for all $m \geq m_0$.*

This type of convergence is denoted by $y'_m \xrightarrow{r} \xi_1$ or alternatively as $r\text{-}\lim_{m\to\infty} y'_m = \xi_1$, where the constant j' is referred to as the roughness degree associated with the sequence $y = \{y'_m\}$.

For a sequence $y = \{y'_m\}$ in the normed space $\mathcal{Y}$, the j'-limit set is defined as

$$\mathrm{LIM}^r_{y'_m} = \{\xi_1 \in \mathbb{X} : y'_m \xrightarrow{r} \xi_1\}.$$

In the case where $y = \{y'_m\}$ is a sequence of real numbers, the j'-limit set can also be expressed as

$$\mathrm{LIM}^r_{y'_m} = [\liminf y'_m - j', \limsup y'_m + j'] \quad \textit{(see [4])}.$$

Definition 2.5. [7] *A sequence $y = y'_m$ is said to be roughly ideal convergent (denoted as r-$\mathtt{I}$-convergent) to a point ξ_1, where $j' \geq 0$ represents the roughness level, if for every $\varepsilon > 0$,*

$$\{m \in \mathcal{N} : \|y'_m - \xi_1\| \geq j' + \varepsilon\} \in \mathtt{I}.$$

Next, we discuss INNLS and the convergence of sequences within this space.

Definition 2.6. *The three-tuple structure $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be an* NNS*, where $\mathcal{Y}$ is a linear space over a field F. Υ, Ω, Γ are called neutrosophic normed space (*NNS*) on $\mathcal{Y} \times (0, \infty)$ and represent the degree of membership and non-membership on $\mathcal{Y} \times (0, 1)$ if the following conditions hold, for every $y, w \in \mathcal{Y}$ and $\lambda_1^*, \lambda_2^* > 0$:*

(1) $\Upsilon(y, \lambda^*) + \Omega(y, \lambda^*) \leq 1$.
(2) $\Upsilon(y, \lambda^*) > 0$.
(3) $\Upsilon(y, \lambda^*) = 1 \Leftrightarrow y = 0$.
(4) $\Upsilon(cy, \lambda^*) = \Upsilon\left(y, \frac{\lambda^*}{|c|}\right)$, *if* $c \neq 0, c \in F$.
(5) $\Upsilon(y, \lambda_1^*) \star \Upsilon(w, \lambda_2^*) \leq \Upsilon(p + w, \lambda_1^* + \lambda_2^*)$.
(6) $\Upsilon(y, .) : (0, \infty) \to [0, 1]$ *is continuous.*
(7) $\lim_{\lambda^*\to\infty} \Upsilon(y, \lambda^*) = 1, \lim_{\lambda^*\to 0} \Upsilon(y, \lambda^*) = 0$.
(8) $\Omega(y, \lambda^*) < 1$.
(9) $\Omega(y, \lambda^*) = 0 \Leftrightarrow y = 0$.
(10) $\Omega(cy, \lambda^*) = \Omega\left(y, \frac{\lambda^*}{|c|}\right)$ *if* $c \neq 0, c \in F$.
(11) $\Omega(y, \lambda_1^*) \diamond \Omega(w, \lambda_2^*) \geq \Omega(y + w, \lambda_1^* + \lambda_2^*)$.
(12) $\Omega(y, .) : (0, \infty) \to [0, 1]$ *is continuous.*
(13) $\lim_{\lambda^*\to\infty} \Omega(y, \lambda^*) = 0, \lim_{\lambda^*\to 0} \Omega(y, \lambda^*) = 1$.
(14) $\Gamma(y, \lambda^*) = 0 \Leftrightarrow y = 0$.
(15) $\Gamma(cy, \lambda^*) = \Omega\left(y, \frac{\lambda^*}{|c|}\right)$ *if* $c \neq 0, c \in F$.
(16) $\Gamma(y, \lambda_1^*) \diamond \Omega(w, \lambda_2^*) \geq \Omega(y + w, \lambda_1^* + \lambda_2^*)$.
(17) $\Gamma(y, .) : (0, \infty) \to [0, 1]$ *is continuous.*
(18) $\lim_{\lambda^*\to\infty} \Gamma(y, \lambda^*) = 0, \lim_{\lambda^*\to 0} \Omega(y, \lambda^*) = 1$.
Then Υ, Ω, Γ are called neutrosophic normed (NN).

Example 2.1. *Let $\mathcal{Y} = \ell^\infty = \left\{y = (y_k)_{k\geq 1} : \|y\|_\infty = \sup_{k\geq 1} |y_k| < \infty\right\}$, be the Banach space of all bounded real sequences equipped with the supremum norm.*

Define

$$a \star b = ab, \qquad a \diamond b = \max\{a, b\},$$

for all $a, b \in [0, 1]$.

For every $y \in \mathcal{Y}$ and $\lambda^* > 0$, define

$$\Upsilon(y, \lambda^*) = \frac{\lambda^*}{\lambda^* + \|y\|_\infty},$$

$$\Omega(y, \lambda^*) = \frac{\|y\|_\infty}{2(\lambda^* + \|y\|_\infty)},$$

$$\Gamma(y, \lambda^*) = \frac{\|y\|_\infty}{2(\lambda^* + \|y\|_\infty)}.$$

We will show that $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ is a neutrosophic normed space.

First,

$$\Upsilon(y, \lambda^*) + \Omega(y, \lambda^*) = \frac{\lambda^* + \frac{1}{2}\|y\|_\infty}{\lambda^* + \|y\|_\infty} \leq 1,$$

which verifies condition (1).

Moreover,

$$\Upsilon(y, \lambda^*) > 0$$

for every $y \in \mathcal{Y}$, and

$$\Upsilon(y, \lambda^*) = 1 \iff \|y\|_\infty = 0 \iff y = 0,$$

so conditions (2) *and* (3) *hold.*

For any nonzero scalar c,

$$\Upsilon(cy, \lambda^*) = \frac{\lambda^*}{\lambda^* + |c|\|y\|_\infty} = \Upsilon\left(y, \frac{\lambda^*}{|c|}\right),$$

establishing condition (4). *Using*

$$\|y + w\|_\infty \leq \|y\|_\infty + \|w\|_\infty,$$

we obtain

$$\Upsilon(y, \lambda_1^*) \star \Upsilon(w, \lambda_2^*) = \frac{\lambda_1^*}{\lambda_1^* + \|y\|_\infty} \cdot \frac{\lambda_2^*}{\lambda_2^* + \|w\|_\infty} \leq \frac{\lambda_1^* + \lambda_2^*}{\lambda_1^* + \lambda_2^* + \|y + w\|_\infty} = \Upsilon(y + w, \lambda_1^* + \lambda_2^*),$$

which verifies (5). *The continuity of Υ follows immediately from its definition. Furthermore,*

$$\lim_{\lambda^* \to \infty} \Upsilon(y, \lambda^*) = 1, \qquad \lim_{\lambda^* \to 0^+} \Upsilon(y, \lambda^*) = 0,$$

for every $y \neq 0$, proving (6) *and* (7). *Similarly,*

$$0 \leq \Omega(y, \lambda^*) < 1,$$

and

$$\Omega(y, \lambda^*) = 0 \iff \|y\|_\infty = 0 \iff y = 0,$$

which establishes (8) *and* (9). *The homogeneity property*

$$\Omega(cy, \lambda^*) = \Omega\left(y, \frac{\lambda^*}{|c|}\right)$$

follows directly from the norm structure, yielding (10). *Using the triangle inequality,*

$$\Omega(y + w, \lambda_1^* + \lambda_2^*) = \frac{\|y + w\|_\infty}{2(\lambda_1^* + \lambda_2^* + \|y + w\|_\infty)} \leq \max\{\Omega(y, \lambda_1^*), \Omega(w, \lambda_2^*)\},$$

which proves (11). *Conditions* (12) *and* (13) *are immediate since*

$$\lim_{\lambda^* \to \infty} \Omega(y, \lambda^*) = 0, \qquad \lim_{\lambda^* \to 0^+} \Omega(y, \lambda^*) = 1 \quad (y \neq 0).$$

Since

$$\Gamma(y, \lambda^*) = \Omega(y, \lambda^*),$$

conditions (14)–(18) *follow exactly as above.*

Hence, all axioms of Definition 2.6 are satisfied. Therefore, $(\ell^{\infty}, \Upsilon, \Omega, \Gamma)$ is a neutrosophic normed space.

Example 2.2. *Let $(\mathcal{Y}, \|.\|)$ be any real normed space. For every $q > 0$ and all $y \in \mathcal{Y}$, define $\Upsilon(y,q) = \frac{q}{q+\|y\|\|}$, $\Omega(y,q) = \frac{y}{q+\|y\|}$, and $\Gamma(y,q) = \frac{y}{q+\|y\|}$. Then, $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ is an INNLS.*

Definition 2.7. *Let $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be an* NNS *with neutrosophic norms $(\Upsilon, \Omega, \Gamma)$. A sequence $y = \{y'_m\}$ in Y is called convergent to $\xi_1 \in Y$ with respect to the norm $(\Upsilon, \Omega, \Gamma)$ if for every $\varepsilon > 0$ and $\lambda^* \in (0,1)$ there exists $m_0 \in \mathcal{N}$ such that $\Upsilon(y'_m - \xi_1; \varepsilon) > 1 - \lambda^*$, and $\Omega(y'_m - \xi_1; \varepsilon) < \lambda^*$, and $\Gamma(y'_m - \xi_1; \varepsilon) < \lambda^*$ for all $m \geq m_0$. It is denoted by $(\Upsilon, \Omega, \Gamma) - \lim_{m\to\infty} y'_m = \xi_1$.*

Definition 2.8. *Consider $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ as an neutrosophic normed space. A sequence $y = \{y'_m\}$ in $\mathcal{Y}$ is considered rough convergent to $\xi'_1 \in \mathcal{Y}$ concerning the norm $(\Upsilon, \Omega, \Gamma)$ for some non-negative real number j' if there exists an $m_0 \in \mathcal{N}$ such that for every $\varepsilon > 0$ and $\lambda^* \in (0,1)$, the following holds*

$$\Upsilon(y'_m - \xi_1; j' + \varepsilon) > 1 - \lambda^* \text{ and } \Omega(y'_m - \xi_1; j' + \varepsilon) < \lambda^* \text{ and } \Gamma(y'_m - \xi_1; j' + \varepsilon) < \lambda^* \text{ for all } m \geq m_0.$$

It is denoted by $y'_m \xrightarrow{j'(\Upsilon,\Omega,\Gamma)} \xi_1$ or $j'(\Upsilon, \Omega, \Gamma) - \lim_{m\to\infty} y'_m = \xi_1$.

Remark 2.1. *For $r = 0$, the rough convergence agrees with the usual convergence for the sequences in an* NNS.

Definition 2.9. *Consider $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ as a neutrosophic normed space. A sequence $y = \{y'_m\}$ in $\mathcal{Y}$ is considered to be roughly statistically convergent to $\xi_1 \in \mathcal{Y}$ concerning the norm $(\Upsilon, \Omega, \Gamma)$ for a non-negative number j' if for all $\varepsilon > 0$ and $\lambda^* \in (0,1)$*

$$\lim_{n\to\infty} \frac{1}{n} \{m \leq n : \Upsilon(y'_m - \xi_1; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^*\} = 0$$

It is represented as $y'_m \xrightarrow{j'-st} \xi_1$ or $j' - st - \lim_{m\to\infty} y'_m = \xi_1$.

Let $st - LIM_y^{j'}$ represent the collection of all rough statistical limit points of the sequence $y = \{y'_m\}$.

3. Main Results

In this section, we formulate the concepts of rough statistical convergence and rough ideal statistical convergence of order $\alpha' \in (0,1]$ in the framework of neutrosophic normed spaces. We then develop several fundamental properties associated with these convergence notions. Throughout the paper, the order parameter α' is taken from the interval $(0,1]$.

Definition 3.1. *Let $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be an neutrosophic normed space. A sequence $y = \{y'_m\}$ in Y is said to be rough statistically convergent of order α' to $\xi_1 \in \mathcal{Y}$ with respect to norm $(\Upsilon, \Omega, \Gamma)$ for some non-negative number j' if for every $\varepsilon > 0$ and $\lambda^* \in (0,1)$.*

$$\lim_{n\to\infty} \frac{1}{n^{\alpha'}} \{m \leq n : \Upsilon(y'_m - \xi_1; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^*\} = 0.$$

It is denoted by $y'_m \xrightarrow{j'-st(\alpha')} \xi_1$ or $j' - \mathrm{st}(\alpha') - \lim_{m\to\infty} y'_m = \xi_1$.
Let $st - LIM_r^{\alpha'} y$ denotes the the set of all rough statistical limit points of order α' of the sequence $y = \{y'_m\}$.

Remark 3.1. *For $r = 0$, the notion rough statistical convergence of order α' agrees with the statistical convergence of order α' for the sequences on an* NNS *and for $\alpha' = 1$, this notion coincides with rough statistical convergence.*

Definition 3.2. *Let $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be an nuetrosophics normed space. A sequence $y = \{y'_m\}$ in $\mathcal{Y}$ is said to be rough ideal statistically convergent of order α' to $\xi_1 \in \mathcal{Y}$ with respect to norm $(\Upsilon, \Omega, \Gamma)$ for some non-negative number j' if for every $\varepsilon > 0$ and $\lambda^* \in (0,1)$.*

$$\left\{ n \in \mathcal{N} : \frac{1}{n^{\alpha'}} \left| \{ m \leq n : \begin{array}{l} \Upsilon(y'_m - \xi_1; j' + \varepsilon) \leq 1 - \lambda^* \\ \text{or } \Omega(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^* \\ \text{or } \Gamma(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^* \} \end{array} \right| \geq \delta^* \right\} \in \mathtt{I},$$

or

$$\delta_I^* \left(\{ m \in \mathcal{N} : \Upsilon \left(y'_m - \xi_1; j' + \varepsilon \right) \leq 1 - \lambda^* \text{ or } \Omega \left(y'_m - \xi_1; j' + \varepsilon \right) \geq \lambda^* \text{ or } \Gamma \left(y'_m - \xi_1; j' + \varepsilon \right) \geq \lambda^* \} \right) = 0,$$

where $\delta_I^*(A) = I - \lim_{n \to \infty} \frac{1}{n^{\alpha'}} |\{ m \leq n : m \in A \}|$ if exists.

It is denoted by $y'_m \xrightarrow{j' - \bar{I} - st(\alpha')} \xi_1$ or $j' - st_I^{\alpha'} - \lim_{m \to \infty} y'_m = \xi_1$.

Let $\mathtt{I} - st - \mathrm{LIM}_{j'}^{\alpha'} y$ denotes the the set of all rough ideal statistical limit points of order α' of the sequence $y = \{y'_m\}$.

Remark 3.2. *For $r = 0$, the notion rough ideal statistical convergence of order a agrees with the ideal statistical convergence of order α' in an* NNS.

The following example shows that there is a sequence which is neither rough ideal convergent nor ideal statistical convergent but it is rough ideal statistical convergent of order α'.

Example 3.1. *Let S be an infinite subset of $\mathcal{N}$ and $\mathtt{I}$ be an ideal such that $S \in I$. Then $S^c = \{a_1 < a_2 < a_3 < \ldots\} \in \mathcal{F}(\mathtt{I})$. Consider a sequence $y = \{y'_m\}_{m \in \mathcal{N}}$ such that*

$$y'_m = \begin{cases} (-1)^m, & \text{if } m \notin S \text{ and } \alpha' = 1, \\ m, & \text{if } m \in S. \end{cases}$$

Then

$$\mathtt{I} - st - \mathrm{LIM}_{j'}^{\alpha'} y = \begin{cases} \phi & j' < 1, \\ [1 - j', j' - 1] & j' \geq 0. \end{cases}$$

In general, a sequence's rough limit might not be unique. Therefore, we consider the rough $\mathtt{I}$-st-limit set of order α' of sequences $y = \{y'_m\}$ with respect to the norm $(\Upsilon, \Omega, \Gamma)$ as

$$\mathtt{I} - st - \mathrm{LIM}_{j'}^{\alpha'} y = \left\{ \xi_1^* \in Y : y'_m \xrightarrow{j' - \mathtt{I} - st(\alpha')} \xi_1^* \right\}.$$

Moreover, sequence $y = \{y'_m\}$ is rough $\mathtt{I}$-st-convergent provided $\mathtt{I}$-st - $\mathrm{LIM}_{j'}^{\alpha'} y \neq \phi$ for some $j' > 0$. In [29], it was observed that for a sequence $y = \{y'_m\}$ of real numbers, the set of rough limit points,

$$\mathrm{LIM}_y^{j'} = [\limsup y - j', \liminf y + j'].$$

In the similar way,

$$\mathtt{I} - st - \mathrm{LIM}_{j'}^{\alpha'} y = [I - st - \limsup y - j', I - st - \liminf y + j'].$$

Definition 3.3. *A sequence $y = \{y'_m\}$ in neutrosophic normed space $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ is* I*-st-bounded of order α'(where $\alpha' \in (0,1]$) if for $\varepsilon > 0, \lambda^* \in (0,1)$ and some $j' > 0$, there exists $\mathcal{H} > 0$ such that*

$$\left\{ n \in \mathcal{N} : \frac{1}{n^{\alpha'}} \left\{ m \leq n : \Upsilon\left(y'_m; \mathcal{H}\right) \leq 1 - \lambda^* \text{ or } \Omega\left(y'_m; \mathcal{H}\right) \geq \lambda^* \text{ or } \Gamma\left(y'_m; \mathcal{H}\right) \geq \lambda^* \right\} \geq \delta^* \right\} \in \mathtt{I}$$

Example 3.2. *Consider the neutrosophic normed space $(\mathbb{R}, \Upsilon, \Omega, \Gamma)$, where*

$$\Upsilon(x;t) = \frac{t}{t+|x|}, \qquad \Omega(x;t) = \frac{|x|}{2(t+|x|)}, \qquad \Gamma(x;t) = \frac{|x|}{2(t+|x|)},$$

for every $x \in \mathbb{R}$ and $t > 0$.

Let $\mathtt{I} = \mathtt{I}_f$ be the ideal of all finite subsets of $\mathbb{N}$, and define the sequence

$$y'_m = \begin{cases} m, & \text{if } m = k^2 \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Now, we show that $\{y'_m\}$ is I*-st-bounded of order $\alpha' = 1$.*

Fix $\lambda^* \in (0,1)$ and choose $\mathcal{H} > 0$. Observe that for all non-square integers m,

$$y'_m = 0,$$

and therefore

$$\Upsilon(y'_m; \mathcal{H}) = 1, \qquad \Omega(y'_m; \mathcal{H}) = 0, \qquad \Gamma(y'_m; \mathcal{H}) = 0.$$

Hence, non-square indices never belong to the set

$$A_n = \left\{ m \leq n : \Upsilon(y'_m; \mathcal{H}) \leq 1 - \lambda^* \text{ or } \Omega(y'_m; \mathcal{H}) \geq \lambda^* \text{ or } \Gamma(y'_m; \mathcal{H}) \geq \lambda^* \right\}.$$

The only possible elements of A_n are square numbers not exceeding n. Since the number of squares less than or equal to n is $\lfloor \sqrt{n} \rfloor$, we obtain

$$|A_n| \leq \sqrt{n}.$$

Consequently,

$$\frac{|A_n|}{n} \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \longrightarrow 0 \qquad (n \to \infty).$$

Let $\delta^* > 0$ be arbitrary. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{|A_n|}{n} < \delta^* \qquad \text{for all } n \geq n_0.$$

Therefore,

$$\left\{ n \in \mathbb{N} : \frac{1}{n}|A_n| \geq \delta^* \right\} \subseteq \{1, 2, \ldots, n_0 - 1\},$$

which is a finite set and hence belongs to $\mathtt{I}_f$.

Thus, $\{y'_m\}$ is I*-statistically bounded of order $\alpha' = 1$.*

Notice that the sequence is not bounded in the ordinary sense since $y'_{k^2} = k^2 \to \infty$. Nevertheless, the unbounded terms occur only on a set of natural density zero, which guarantees I*-statistical boundedness.*

We first establish the existence of a minimal roughness parameter for which the I-statistical limit set of order α' is nonempty.

Theorem 3.1. *Let $\{y'_m\}$ be a sequence in a neutrosophic normed space $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ such that*

$$\mathtt{I}\text{-}st\text{-}\mathrm{LIM}_{j'}^{\alpha'}(y) \neq \varnothing \quad \text{for some } j' > 0.$$

Define

$$j_0 = \inf \left\{ j' > 0 : \mathtt{I}\text{-}st\text{-}\mathrm{LIM}_{j'}^{\alpha'}(y) \neq \varnothing \right\}.$$

Then

$$\mathtt{I}\text{-}st\text{-}\operatorname{LIM}_{j_0}^{\alpha'}(y) \neq \varnothing.$$

Proof. Since $\mathtt{I}\text{-}st\text{-}\operatorname{LIM}_{j'}^{\alpha'}(y) \neq \varnothing$ for some $j' > 0$, there exists $\xi_1 \in \mathcal{Y}$ such that for every $\varepsilon > 0$ and $\lambda^* \in (0,1)$,

$$\left\{ n \in \mathcal{N} : \frac{1}{n^{\alpha'}} \Big\{ m \leq n : \Upsilon(y'_m - \xi_1; j' + \varepsilon) \leq 1 - \lambda^* \right.$$
$$\text{or } \Omega(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^*$$
$$\left. \left. \text{or } \Gamma(y'_m - \xi_1; j' + \varepsilon) \geq \lambda^* \Big\} \geq \delta^* \right\} \in \mathtt{I}.$$

Hence, the set $\mathcal{J} = \{j' > 0 : \ \mathtt{I}\text{-}st\text{-}\operatorname{LIM}_{j'}^{\alpha'}(y) \neq \varnothing\}$ is nonempty and bounded below by 0. Therefore, $j_0 = \inf \mathcal{J}$ exists.

By definition of infimum, there exists a decreasing sequence $\{j_k\} \subset \mathcal{J}$ such that

$$j_k > j_0 \quad \text{for all } k, \qquad \text{and} \qquad j_k \downarrow j_0.$$

For each k, choose

$$\xi_k \in \mathtt{I}\text{-}st\text{-}\operatorname{LIM}_{j_k}^{\alpha'}(y).$$

Then for every $\varepsilon > 0$ and $\lambda^* \in (0,1)$. For each $k \in \mathbb{N}$, define a set

$$E_k = \left\{ n \in \mathcal{N} : \frac{1}{n^{\alpha'}} \Big\{ m \leq n : \Upsilon(y'_m - \xi_k; j_k + \varepsilon) \leq 1 - \lambda^* \right.$$
$$\text{or } \Omega(y'_m - \xi_k; j_k + \varepsilon) \geq \lambda^*$$
$$\left. \left. \text{or } \Gamma(y'_m - \xi_k; j_k + \varepsilon) \geq \lambda^* \Big\} \geq \delta^* \right\} \in \mathtt{I}.$$

Similarly, since $\{y'_m\}$ is $\mathtt{I}$-st-bounded of order α', there exists $\mathcal{H} > 0$ such that

$$E_0 = \left\{ n \in \mathcal{N} : \frac{1}{n^{\alpha'}} \Big\{ m \leq n : \Upsilon(y'_m; \mathcal{H}) \leq 1 - \lambda^* \right.$$
$$\text{or } \Omega(y'_m; \mathcal{H}) \geq \lambda^*$$
$$\left. \left. \text{or } \Gamma(y'_m; \mathcal{H}) \geq \lambda^* \Big\} \geq \delta^* \right\} \in \mathtt{I}.$$

Since $\mathtt{I}$ is admissible, the set

$$\mathcal{N} \setminus (E_k \cup E_0) \neq \varnothing.$$

Choose $m \in \mathcal{N} \setminus (E_k \cup E_0)$. Then we have

$$\begin{aligned} \Upsilon(\xi_k; j_k + \varepsilon + \mathcal{H}) &= \Upsilon((\xi_k - y'_m) + y'_m; j_k + \varepsilon + \mathcal{H}) \\ &\geq \min\{\Upsilon(y'_m - \xi_k; j_k + \varepsilon), \Upsilon(y'_m; \mathcal{H})\} \\ &> 1 - \lambda^*, \end{aligned}$$

and similarly,

$$\begin{aligned} \Omega(\xi_k; j_k + \varepsilon + \mathcal{H}) &\leq \max\{\Omega(y'_m - \xi_k; j_k + \varepsilon), \Omega(y'_m; \mathcal{H})\} \\ &< \lambda^*, \\ \Gamma(\xi_k; j_k + \varepsilon + \mathcal{H}) &\leq \max\{\Gamma(y'_m - \xi_k; j_k + \varepsilon), \Gamma(y'_m; \mathcal{H})\} \\ &< \lambda^*. \end{aligned}$$

Hence, for every $k \in \mathbb{N}$,

$$\xi_k \in B\big(0,\, j_k + \varepsilon + \mathcal{H}\big).$$

Therefore, $\{\xi_k\}$ is bounded in $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$.

By completeness, there exists a subsequence $\{\xi_{k_\ell}\}$ such that

$$\xi_{k_\ell} \to \xi \quad \text{in } (\mathcal{Y}, \Upsilon, \Omega, \Gamma).$$

That is, for every $\varepsilon > 0$ and $\lambda^* \in (0,1)$, there exists ℓ_0 such that

$$\Upsilon(\xi_{k_\ell} - \xi; \varepsilon) > 1 - \lambda^*, \quad \Omega(\xi_{k_\ell} - \xi; \varepsilon) < \lambda^*, \quad \Gamma(\xi_{k_\ell} - \xi; \varepsilon) < \lambda^*$$

for all $\ell \geq \ell_0$.

For a fix $\varepsilon > 0$ and $\lambda^* \in (0,1)$, we obtain

$$\Upsilon(y'_m - \xi; j_0 + \varepsilon) \geq \limsup_{\ell \to \infty} \Upsilon(y'_m - \xi_{k_\ell}; j_{k_\ell} + \varepsilon),$$

and similarly,

$$\Omega(y'_m - \xi; j_0 + \varepsilon) \leq \liminf_{\ell \to \infty} \Omega(y'_m - \xi_{k_\ell}; j_{k_\ell} + \varepsilon),$$
$$\Gamma(y'_m - \xi; j_0 + \varepsilon) \leq \liminf_{\ell \to \infty} \Gamma(y'_m - \xi_{k_\ell}; j_{k_\ell} + \varepsilon).$$

Therefore,

$$\Bigg\{ n \in \mathcal{N} : \frac{1}{n^{\alpha'}} \Big\{ m \leq n : \Upsilon(y'_m - \xi; j_0 + \varepsilon) \leq 1 - \lambda^* \\ \text{or } \Omega(y'_m - \xi; j_0 + \varepsilon) \geq \lambda^* \\ \text{or } \Gamma(y'_m - \xi; j_0 + \varepsilon) \geq \lambda^* \Big\} \geq \delta^* \Bigg\} \in \mathtt{I}.$$

Thus,

$$\xi \in \mathtt{I}\text{-}st\text{-}\operatorname{LIM}^{\alpha'}_{j_0}(y),$$

which implies

$$\mathtt{I}\text{-}st\text{-}\operatorname{LIM}^{\alpha'}_{j_0}(y) \neq \varnothing.$$

This completes the proof. □

We now prove that rough I-statistical convergence of order α' is stable under small perturbations.

Theorem 3.2. *Let $\{y'_m\}$ and $\{z'_m\}$ be sequences in a neutrosophic normed space such that*

$$\delta^*_I\big(\{m : \Upsilon(y'_m - z'_m; \varepsilon) \leq 1 - \lambda^*\}\big) = 0.$$

Then, for any $j' > 0$,

$$y'_m \xrightarrow{j'-\mathtt{I}\text{-}st(\alpha')} \xi \quad \Longleftrightarrow \quad z'_m \xrightarrow{j'-\mathtt{I}\text{-}st(\alpha')} \xi.$$

Proof. We establish the forward implication only, since the converse follows by symmetry.

Assume that

$$y'_m \xrightarrow{j'-\mathtt{I}\text{-}st(\alpha')} \xi.$$

Let $\varepsilon > 0$ and $\lambda^* \in (0,1)$ be arbitrary. By the definition of j'-I-statistical convergence of order α', we have

$$A = \Big\{ m \in \mathbb{N} : \Upsilon(y'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \Big\} \in \mathtt{I}.$$

On the other hand, by the assumption

$$\delta^*_I\Big(\{m : \Upsilon(y'_m - z'_m; \varepsilon) \leq 1 - \lambda^*\}\Big) = 0,$$

and similarly for the indeterminacy and falsity functions, it follows that the set

$$B = \Big\{ m \in \mathbb{N} : \Upsilon(y'_m - z'_m; \varepsilon) \le 1 - \lambda^* \text{ or } \Omega(y'_m - z'_m; \varepsilon) \ge \lambda^* \text{ or } \Gamma(y'_m - z'_m; \varepsilon) \ge \lambda^* \Big\} \in \mathtt{I}.$$

Since $\mathtt{I}$ is an ideal, it is closed under finite unions. Therefore,

$$A \cup B \in \mathtt{I}.$$

Now, consider any index $m \notin A \cup B$. Then $m \notin A$ and $m \notin B$, so that

$$\Upsilon(y'_m - \xi; j' + \varepsilon) > 1 - \lambda^*, \qquad \Omega(y'_m - \xi; j' + \varepsilon) < \lambda^*, \qquad \Gamma(y'_m - \xi; j' + \varepsilon) < \lambda^*,$$

and

$$\Upsilon(y'_m - z'_m; \varepsilon) > 1 - \lambda^*, \qquad \Omega(y'_m - z'_m; \varepsilon) < \lambda^*, \qquad \Gamma(y'_m - z'_m; \varepsilon) < \lambda^*.$$

Using the neutrosophic norm inequality, we get

$$\Upsilon(x + y, s + t) \ge \min\{\Upsilon(x, s), \Upsilon(y, t)\},$$

with

$$x = y'_m - \xi, \qquad y = z'_m - y'_m,$$

and noting that

$$z'_m - \xi = (y'_m - \xi) + (z'_m - y'_m),$$

we obtain

$$\begin{aligned} \Upsilon(z'_m - \xi; j' + 2\varepsilon) &= \Upsilon\big((y'_m - \xi) + (z'_m - y'_m), (j' + \varepsilon) + \varepsilon\big) \\ &\ge \min\Big\{ \Upsilon(y'_m - \xi; j' + \varepsilon), \Upsilon(z'_m - y'_m; \varepsilon) \Big\}. \end{aligned}$$

Since

$$\Upsilon(z'_m - y'_m; \varepsilon) = \Upsilon(y'_m - z'_m; \varepsilon),$$

it follows that

$$\Upsilon(z'_m - \xi; j' + 2\varepsilon) > 1 - \lambda^*.$$

Similarly,

$$\Omega(x + y, s + t) \le \max\{\Omega(x, s), \Omega(y, t)\},$$

we obtain

$$\begin{aligned} \Omega(z'_m - \xi; j' + 2\varepsilon) &\le \max\Big\{ \Omega(y'_m - \xi; j' + \varepsilon), \Omega(z'_m - y'_m; \varepsilon) \Big\} \\ &< \lambda^*. \end{aligned}$$

Again,

$$\Gamma(x + y, s + t) \le \max\{\Gamma(x, s), \Gamma(y, t)\},$$

we have

$$\begin{aligned} \Gamma(z'_m - \xi; j' + 2\varepsilon) &\le \max\Big\{ \Gamma(y'_m - \xi; j' + \varepsilon), \Gamma(z'_m - y'_m; \varepsilon) \Big\} \\ &< \lambda^*. \end{aligned}$$

Hence, for every $m \notin A \cup B$,

$$\Upsilon(z'_m - \xi; j' + 2\varepsilon) > 1 - \lambda^*,$$
$$\Omega(z'_m - \xi; j' + 2\varepsilon) < \lambda^*,$$

and

$$\Gamma(z'_m - \xi; j' + 2\varepsilon) < \lambda^*.$$

Consequently,

$$\Big\{ m \in \mathbb{N} : \Upsilon(z'_m - \xi; j' + 2\varepsilon) \le 1 - \lambda^* \text{ or } \Omega(z'_m - \xi; j' + 2\varepsilon) \ge \lambda^* \text{ or } \Gamma(z'_m - \xi; j' + 2\varepsilon) \ge \lambda^* \Big\}$$
$$\subseteq A \cup B.$$

Since $A \cup B \in \mathtt{I}$, we conclude that

$$\Big\{m : \Upsilon(z'_m - \xi; j' + 2\varepsilon) \le 1 - \lambda^* \text{ or } \Omega(z'_m - \xi; j' + 2\varepsilon) \ge \lambda^* \text{ or } \Gamma(z'_m - \xi; j' + 2\varepsilon) \ge \lambda^*\Big\} \in \mathtt{I}.$$

As $\varepsilon > 0$ and $\lambda^* \in (0, 1)$ were arbitrary, the definition of j'-$\mathtt{I}$-statistical convergence of order α' yields

$$z'_m \xrightarrow{j'-\mathtt{I}\text{-}st(\alpha')} \xi.$$

The converse implication is proved in exactly the same manner by interchanging the roles of the sequences $\{y'_m\}$ and $\{z'_m\}$. Therefore,

$$y'_m \xrightarrow{j'-\mathtt{I}\text{-}st(\alpha')} \xi \quad \Longleftrightarrow \quad z'_m \xrightarrow{j'-\mathtt{I}\text{-}st(\alpha')} \xi.$$

This completes the proof. □

Next, we show that if the intersection of all rough $\mathtt{I}$-statistical limit sets of order α' is nonempty, then the limit point is unique.

Theorem 3.3. *If $\bigcap_{j'>0} \mathtt{I}\text{-}st\text{-}\mathrm{LIM}^{\alpha'}_{j'}(y) \neq \varnothing$, then the intersection consists of exactly one point.*

Proof. Assume, to the contrary, that there exist two distinct points $\xi_1, \xi_2 \in \mathcal{Y}$ such that

$$\xi_1 \neq \xi_2 \quad \text{and} \quad \xi_1, \xi_2 \in \bigcap_{j'>0} \mathtt{I}\text{-}st\text{-}\mathrm{LIM}^{\alpha'}_{j'}(y).$$

Fix $\varepsilon > 0$ and $\lambda^* \in (0, 1)$. Since $\xi_1 \in \mathtt{I}\text{-}st\text{-}\mathrm{LIM}^{\alpha'}_{j'}(y)$, there exists a set $A_{j'} \in \mathtt{I}$ such that for all $m \notin A_{j'}$,

$$\Upsilon(y'_m - \xi_1; j' + \varepsilon) > 1 - \lambda^*, \quad \Omega(y'_m - \xi_1; j' + \varepsilon) < \lambda^*, \quad \Gamma(y'_m - \xi_1; j' + \varepsilon) < \lambda^*. \tag{1}$$

Similarly, since $\xi_2 \in \mathtt{I}\text{-}st\text{-}\mathrm{LIM}^{\alpha'}_{j'}(y)$, there exists a set $B_{j'} \in \mathtt{I}$ such that for all $m \notin B_{j'}$,

$$\Upsilon(y'_m - \xi_2; j' + \varepsilon) > 1 - \lambda^*, \quad \Omega(y'_m - \xi_2; j' + \varepsilon) < \lambda^*, \quad \Gamma(y'_m - \xi_2; j' + \varepsilon) < \lambda^*. \tag{2}$$

Since $\mathtt{I}$ is an admissible ideal, $A_{j'} \cup B_{j'} \in \mathtt{I}$. Choose $m \notin A_{j'} \cup B_{j'}$. Then combining (1) and (2), we obtain

$$\Upsilon(\xi_1 - \xi_2; 2j' + 2\varepsilon) \ge \min\{\Upsilon(y'_m - \xi_1; j' + \varepsilon), \Upsilon(y'_m - \xi_2; j' + \varepsilon)\} > 1 - \lambda^*,$$

$$\Omega(\xi_1 - \xi_2; 2j' + 2\varepsilon) \le \max\{\Omega(y'_m - \xi_1; j' + \varepsilon), \Omega(y'_m - \xi_2; j' + \varepsilon)\} < \lambda^*,$$

$$\Gamma(\xi_1 - \xi_2; 2j' + 2\varepsilon) \le \max\{\Gamma(y'_m - \xi_1; j' + \varepsilon), \Gamma(y'_m - \xi_2; j' + \varepsilon)\} < \lambda^*.$$

Since $j' > 0$ is arbitrary, letting $j' \to 0^+$, we obtain

$$\Upsilon(\xi_1 - \xi_2; 2\varepsilon) = 1, \quad \Omega(\xi_1 - \xi_2; 2\varepsilon) = 0, \quad \Gamma(\xi_1 - \xi_2; 2\varepsilon) = 0.$$

this implies that

$$\xi_1 - \xi_2 = 0,$$

that is, $\xi_1 = \xi_2$, which contradicts the assumption $\xi_1 \neq \xi_2$.

Hence, the intersection $\bigcap_{j'>0} \mathtt{I}\text{-}st\text{-}\mathrm{LIM}^{\alpha'}_{j'}(y)$ contains exactly one point. □

We next characterize rough $\mathtt{I}$-statistical convergence of order α' via subsequences indexed by filter sets.

Theorem 3.4. *A sequence $\{y'_m\}$ in $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ is rough $\mathtt{I}$-statistically convergent of order α' if and only if it admits a subsequence indexed by a set $M \in \mathcal{F}(\mathtt{I})$ which is rough statistically convergent of order α'.*

Proof. Assume that the sequence $\{y'_m\}$ is rough $\mathtt{I}$-statistically convergent of order α' to ξ with roughness $j' > 0$. Then, by definition, for every $\varepsilon > 0$ and $\lambda^* \in (0,1)$,

$$\delta_I^*\Big(\{m \in \mathcal{N} : \Upsilon(y'_m - \xi; j' + \varepsilon) \le 1 - \lambda^* \text{ or } \Omega(y'_m - \xi; j' + \varepsilon) \ge \lambda^* \text{ or } \Gamma(y'_m - \xi; j' + \varepsilon) \ge \lambda^*\}\Big) = 0. \tag{3}$$

Define the set

$$M = \{m \in \mathcal{N} : \Upsilon(y'_m - \xi; j' + \varepsilon) > 1 - \lambda^*, \ \Omega(y'_m - \xi; j' + \varepsilon) < \lambda^*, \ \Gamma(y'_m - \xi; j' + \varepsilon) < \lambda^*\}.$$

From (3), we have

$$\mathcal{N} \setminus M \in \mathtt{I},$$

and hence $M \in \mathcal{F}(\mathtt{I})$.

Now consider the subsequence $\{y'_m\}_{m \in M}$. For each $n \in \mathcal{N}$, define

$$M_n = \{m \le n : \ m \in M\}.$$

Then, for all $m \in M_n$, the inequalities

$$\Upsilon(y'_m - \xi; j' + \varepsilon) > 1 - \lambda^*, \quad \Omega(y'_m - \xi; j' + \varepsilon) < \lambda^*, \quad \Gamma(y'_m - \xi; j' + \varepsilon) < \lambda^*$$

hold.

Hence,

$$\frac{1}{n^{\alpha'}}\Big|\{m \le n : \ m \in M \text{ and } \Upsilon(y'_m - \xi; j' + \varepsilon) \le 1 - \lambda^* \text{ or } \Omega(y'_m - \xi; j' + \varepsilon) \ge \lambda^* \text{ or } \Gamma(y'_m - \xi; j' + \varepsilon) \ge \lambda^*\}\Big| = 0.$$

Therefore,

$$\lim_{n \to \infty} \frac{1}{n^{\alpha'}}\Big|\{m \le n : \ m \in M \text{ and } y'_m \notin B(\xi, j' + \varepsilon, \lambda^*)\}\Big| = 0,$$

which shows that the subsequence $\{y'_m\}_{m \in M}$ is rough statistically convergent of order α' to ξ.

Conversely, suppose that there exists a set $M \in \mathcal{F}(\mathtt{I})$ such that the subsequence $\{y'_m\}_{m \in M}$ is rough statistically convergent of order α' to ξ with roughness $j' > 0$.

Then, for every $\varepsilon > 0$ and $\lambda^* \in (0,1)$,

$$\lim_{n \to \infty} \frac{1}{n^{\alpha'}}\Big|\{m \le n : \ m \in M \text{ and } \Upsilon(y'_m - \xi; j' + \varepsilon) \le 1 - \lambda^* \text{ or } \Omega(y'_m - \xi; j' + \varepsilon) \ge \lambda^* \text{ or } \Gamma(y'_m - \xi; j' + \varepsilon) \ge \lambda^*\}\Big| = 0.$$

Since $M \in \mathcal{F}(\mathtt{I})$, we have $\mathcal{N} \setminus M \in \mathtt{I}$. Let

$$A = \{m \in \mathcal{N} : \ \Upsilon(y'_m - \xi; j' + \varepsilon) \le 1 - \lambda^* \text{ or } \Omega(y'_m - \xi; j' + \varepsilon) \ge \lambda^* \text{ or } \Gamma(y'_m - \xi; j' + \varepsilon) \ge \lambda^*\}.$$

Then

$$A \subseteq (\mathcal{N} \setminus M) \cup (A \cap M).$$

By subadditivity of δ_I^*,

$$\delta_I^*(A) \le \delta_I^*(\mathcal{N} \setminus M) + \delta_I^*(A \cap M).$$

Since $\mathcal{N} \setminus M \in \mathtt{I}$, $\delta_I^*(\mathcal{N} \setminus M) = 0$, and $\delta_I^*(A \cap M) = 0$. Hence,

$$\delta_I^*(A) = 0.$$

Thus,

$$\delta_I^*\Big(\{m \in \mathcal{N} : \Upsilon(y'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(y'_m - \xi; j' + \varepsilon) \geq \lambda^*\}\Big) = 0.$$

This proves that $\{y'_m\}$ is rough $\mathtt{I}$-statistically convergent of order α' to ξ. □

Finally, we present a decomposition characterization of rough $\mathtt{I}$-statistical convergence of order α'.

Theorem 3.5. *A sequence $\{y'_m\}$ is rough $\mathtt{I}$-statistically convergent of order α' if and only if it can be written as $y'_m = u_m + v_m$, where $\{u_m\}$ is $\mathtt{I}$-statistically convergent of order α' and $\{v_m\}$ is $\mathtt{I}$-statistically bounded by j'.*

Proof. Assume that $\{y'_m\}$ is rough $\mathtt{I}$-statistically convergent of order α' to ξ with roughness $j' > 0$. Define sequences $\{u_m\}$ and $\{v_m\}$ by

$$u_m = \xi, \qquad v_m = y'_m - \xi \quad (m \in \mathcal{N}).$$

We first show that $\{u_m\}$ is $\mathtt{I}$-statistically convergent of order α' to ξ. Since $u_m - \xi = 0$ for all m, we have for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$,

$$\Upsilon(u_m - \xi; \varepsilon) = 1, \quad \Omega(u_m - \xi; \varepsilon) = 0, \quad \Gamma(u_m - \xi; \varepsilon) = 0,$$

which immediately implies

$$\delta_{\mathtt{I}}^{\alpha'}\Big(\{m : \Upsilon(u_m - \xi; \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(u_m - \xi; \varepsilon) \geq \lambda^* \text{ or } \Gamma(u_m - \xi; \varepsilon) \geq \lambda^*\}\Big) = 0.$$

Hence, $\{u_m\}$ is $\mathtt{I}$-statistically convergent of order α' to ξ.

Next, we show that $\{v_m\}$ is $\mathtt{I}$-statistically bounded of order α' by j'. Since y'_m is rough $\mathtt{I}$-statistically convergent to ξ, for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$,

$$\delta_{\mathtt{I}}^{\alpha'}\Big(\{m : \Upsilon(v_m; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(v_m; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(v_m; j' + \varepsilon) \geq \lambda^*\}\Big) = 0.$$

This proves that $\{v_m\}$ is $\mathtt{I}$-statistically bounded of order α' with bound j'.

Conversely, assume that $y'_m = u_m + v_m$, where $\{u_m\}$ is $\mathtt{I}$-statistically convergent of order α' to ξ and $\{v_m\}$ is $\mathtt{I}$-statistically bounded of order α' with bound j'.

Fix $\varepsilon > 0$ and $\lambda^* \in (0, 1)$. Define

$$A = \{m : \Upsilon(u_m - \xi; \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(u_m - \xi; \varepsilon) \geq \lambda^* \text{ or } \Gamma(u_m - \xi; \varepsilon) \geq \lambda^*\},$$

$$B = \{m : \Upsilon(v_m; j') \leq 1 - \lambda^* \text{ or } \Omega(v_m; j') \geq \lambda^* \text{ or } \Gamma(v_m; j') \geq \lambda^*\}.$$

Then $A, B \in \mathtt{I}$. Since $\mathtt{I}$ is admissible, $A \cup B \in \mathtt{I}$.

Let $m \notin A \cup B$. Then we obtain

$$\Upsilon(y'_m - \xi; j' + \varepsilon) \geq \min\{\Upsilon(u_m - \xi; \varepsilon), \Upsilon(v_m; j')\} > 1 - \lambda^*,$$

$$\Omega(y'_m - \xi; j' + \varepsilon) \leq \max\{\Omega(u_m - \xi; \varepsilon), \Omega(v_m; j')\} < \lambda^*,$$

$$\Gamma(y'_m - \xi; j' + \varepsilon) \leq \max\{\Gamma(u_m - \xi; \varepsilon), \Gamma(v_m; j')\} < \lambda^*.$$

Therefore,

$$\delta_{\mathtt{I}}^{\alpha'}\Big(\{m : \Upsilon(y'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \text{ or } \Omega(y'_m - \xi; j' + \varepsilon) \geq \lambda^* \text{ or } \Gamma(y'_m - \xi; j' + \varepsilon) \geq \lambda^*\}\Big) = 0,$$

which shows that $\{y'_m\}$ is rough $\mathtt{I}$-statistically convergent of order α' to ξ with roughness j'. □

3.1. **Application to Fixed Point Iteration in Neutrosophic Normed Spaces.** Let $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be a neutrosophic normed space and let $T : \mathcal{Y} \to \mathcal{Y}$ be a self-mapping having a fixed point ξ, that is, $T(\xi) = \xi$. Consider the Picard iterative process $y'_{m+1} = T(y'_m), \qquad m \in \mathbb{N}$. In practical computations, measurements and rounding errors are unavoidable. Hence, the generated sequence may take the form $z'_m = y'_m + e_m$, where e_m denotes the error sequence.

Suppose that $y'_m \xrightarrow{\mathtt{I}-st(\alpha')} \xi$

and that the error sequence is $\mathtt{I}$-statistically bounded of order α' by some roughness degree $j' > 0$.

Then, by Theorem 3.5, the perturbed iteration sequence

$$z'_m = y'_m + e_m$$

is rough $\mathtt{I}$-statistically convergent of order α' to the same fixed point ξ.

Consequently, even in the presence of neutrosophic uncertainty and statistical perturbations, the iterative algorithm preserves convergence up to the roughness level j'. Therefore, rough $\mathtt{I}$-statistical convergence provides a quantitative measure of stability for fixed-point algorithms in neutrosophic environments.

This application is particularly useful in numerical methods, optimization, artificial intelligence, and decision-making models where exact convergence cannot be guaranteed because of incomplete, indeterminate, or inconsistent information.

Corollary 3.1. *Let $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be a neutrosophic normed space and let $y' * m$ be generated by the Picard iteration*

$$y' * m + 1 = T(y'_m),$$

where T possesses a fixed point ξ.

If y'_m is $\mathtt{I}$-statistically convergent of order α' to ξ and the perturbation sequence e_m is $\mathtt{I}$-statistically bounded of order α' by j', then the perturbed sequence

$$z'_m = y'_m + e_m$$

is rough $\mathtt{I}$-statistically convergent of order α' to ξ with roughness degree j'.

Proof. Since $\{y'_m\}$ is $\mathtt{I}$-statistically convergent of order α' to the fixed point ξ, it follows that for every $\varepsilon > 0$ and $\lambda^* \in (0, 1)$,

$$\delta_I^*\Big(\big\{m \in \mathbb{N} : \Upsilon(y'_m - \xi; \varepsilon) \le 1 - \lambda^* \text{ or } \Omega(y'_m - \xi; \varepsilon) \ge \lambda^* \text{ or } \Gamma(y'_m - \xi; \varepsilon) \ge \lambda^*\big\}\Big) = 0. \tag{1}$$

Furthermore, since $\{e_m\}$ is $\mathtt{I}$-statistically bounded of order α' by j', it follows that for every $\lambda^* \in (0, 1)$,

$$\delta_I^*\Big(\big\{m \in \mathbb{N} : \Upsilon(e_m; j') \le 1 - \lambda^* \text{ or } \Omega(e_m; j') \ge \lambda^* \text{ or } \Gamma(e_m; j') \ge \lambda^*\big\}\Big) = 0. \tag{2}$$

For fixed $\varepsilon > 0$ and $\lambda^* \in (0, 1)$, we define

$$A = \Big\{m \in \mathbb{N} : \Upsilon(y'_m - \xi; \varepsilon) \le 1 - \lambda^* \text{ or } \Omega(y'_m - \xi; \varepsilon) \ge \lambda^* \text{ or } \Gamma(y'_m - \xi; \varepsilon) \ge \lambda^*\Big\},$$

and

$$B = \Big\{ m \in \mathbb{N} : \Upsilon(e_m; j') \leq 1 - \lambda^* \\ \text{or } \Omega(e_m; j') \geq \lambda^* \\ \text{or } \Gamma(e_m; j') \geq \lambda^* \Big\}.$$

By (1) and (2), we have

$$\delta_I^*(A) = 0, \qquad \delta_I^*(B) = 0.$$

Since δ_I^* is subadditive, we obtain

$$\delta_I^*(A \cup B) = 0.$$

Now, let

$$m \notin A \cup B.$$

Then,

$$\Upsilon(y'_m - \xi; \varepsilon) > 1 - \lambda^*, \qquad \Omega(y'_m - \xi; \varepsilon) < \lambda^*, \qquad \Gamma(y'_m - \xi; \varepsilon) < \lambda^*,$$

and

$$\Upsilon(e_m; j') > 1 - \lambda^*, \qquad \Omega(e_m; j') < \lambda^*, \qquad \Gamma(e_m; j') < \lambda^*.$$

Since

$$z'_m - \xi = (y'_m - \xi) + e_m,$$

the neutrosophic triangle inequality yields

$$\begin{aligned} \Upsilon(z'_m - \xi; j' + \varepsilon) &= \Upsilon\big((y'_m - \xi) + e_m, \varepsilon + j'\big) \\ &\geq \min\Big\{ \Upsilon(y'_m - \xi; \varepsilon), \Upsilon(e_m; j') \Big\} \\ &> 1 - \lambda^*. \end{aligned}$$

Similarly,

$$\begin{aligned} \Omega(z'_m - \xi; j' + \varepsilon) &\leq \max\Big\{ \Omega(y'_m - \xi; \varepsilon), \Omega(e_m; j') \Big\} \\ &< \lambda^*, \end{aligned}$$

and

$$\begin{aligned} \Gamma(z'_m - \xi; j' + \varepsilon) &\leq \max\Big\{ \Gamma(y'_m - \xi; \varepsilon), \Gamma(e_m; j') \Big\} \\ &< \lambda^*. \end{aligned}$$

Consequently,

$$C = \Big\{ m \in \mathbb{N} : \Upsilon(z'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \\ \text{or } \Omega(z'_m - \xi; j' + \varepsilon) \geq \lambda^* \\ \text{or } \Gamma(z'_m - \xi; j' + \varepsilon) \geq \lambda^* \Big\} \subseteq A \cup B.$$

Therefore,

$$\delta_I^*(C) \leq \delta_I^*(A \cup B) = 0.$$

Hence,

$$\delta_I^*\Big(\big\{ m \in \mathbb{N} : \Upsilon(z'_m - \xi; j' + \varepsilon) \leq 1 - \lambda^* \\ \text{or } \Omega(z'_m - \xi; j' + \varepsilon) \geq \lambda^* \\ \text{or } \Gamma(z'_m - \xi; j' + \varepsilon) \geq \lambda^* \big\} \Big) = 0.$$

Since $\varepsilon > 0$ and $\lambda^* \in (0,1)$ were arbitrary, by the definition of rough $\mathtt{I}$-statistical convergence of order α', we conclude that

$$z'_m \xrightarrow{j'-\mathtt{I}\text{-}st(\alpha')} \xi.$$

Therefore, the perturbed Picard sequence $\{z'_m\}$ is rough $\mathtt{I}$-statistically convergent of order α' to the fixed point ξ with roughness degree j'. □

Example 3.3. *Consider the neutrosophic normed space* $(\mathbb{R}, \Upsilon, \Omega, \Gamma)$, *where*

$$\Upsilon(x;t) = \frac{t}{t+|x|}, \qquad \Omega(x;t) = \frac{|x|}{2(t+|x|)}, \qquad \Gamma(x;t) = \frac{|x|}{2(t+|x|)}$$

for every $x \in \mathbb{R}$ *and* $t > 0$.

Let $\mathtt{I} = \mathtt{I}_f$ *be the ideal of all finite subsets of* $\mathbb{N}$ *and let*

$$T(x) = \frac{x}{2}.$$

Clearly, T *has the unique fixed point*

$$\xi = 0.$$

Starting from $y'_1 = 1$, *consider the Picard iteration*

$$y'_{m+1} = T(y'_m) = \frac{y'_m}{2}.$$

Then,

$$y'_m = \frac{1}{2^{m-1}}.$$

Since

$$\lim_{m\to\infty} y'_m = 0,$$

it follows that

$$y'_m \xrightarrow{\mathtt{I}_f\text{-}st(1)} 0.$$

Now, we define a perturbation sequence

$$e_m = \begin{cases} 1, & \text{if } m = k^2 \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$j' = 1.$$

The set of indices for which $e_m \neq 0$ *is*

$$A = \{1, 4, 9, 16, \ldots\}.$$

Since

$$|A \cap \{1, \ldots, n\}| = \lfloor \sqrt{n} \rfloor,$$

we obtain

$$\lim_{n\to\infty} \frac{\lfloor \sqrt{n} \rfloor}{n} = 0.$$

Hence, $\{e_m\}$ *is* $\mathtt{I}_f$*-statistically bounded of order* $\alpha' = 1$ *by* $j' = 1$.

Define the perturbed sequence

$$z'_m = y'_m + e_m.$$

Then,

$$z'_m = \begin{cases} \dfrac{1}{2^{m-1}} + 1, & \text{if } m = k^2, \\ \dfrac{1}{2^{m-1}}, & \text{otherwise.} \end{cases}$$

Observe that

$$z'_{k^2} = 1 + \frac{1}{2^{k^2-1}} \longrightarrow 1,$$

whereas

$$z'_m = \frac{1}{2^{m-1}} \longrightarrow 0$$

for non-square indices. Consequently, the sequence $\{z'_m\}$ *is not ordinarily convergent.*

However, the exceptional terms occur only on the set of square numbers, which has natural density zero. Therefore, by the above corollary,

$$z'_m \xrightarrow{1-\mathtt{I}_f\text{-}st(1)} 0.$$

Hence,

$$0 \in \mathtt{I}_f\text{-}st\text{-}\mathrm{LIM}_1^1(z).$$

Thus, although the perturbations destroy ordinary convergence, the perturbed Picard iteration remains rough I*-statistically convergent to the fixed point* $\xi = 0$ *with roughness degree* 1*.*

3.2. **Application to Stability Analysis of Neutrosophic Dynamical Systems.** Consider a discrete-time neutrosophic dynamical system

$$x_{m+1} = T(x_m) + e_m,$$

where $T : \mathcal{Y} \to \mathcal{Y}$ is a system operator and $\{e_m\}$ denotes external disturbances.

In practical situations, disturbances may contain rare but arbitrarily large impulses caused by transmission failures, abrupt environmental changes, or incomplete information. Such impulses generally prevent ordinary convergence of trajectories.

Suppose that the disturbance sequence satisfies

$$\delta_I^*\Big(\{m : \|e_m\| > \varepsilon\}\Big) = 0$$

for every $\varepsilon > 0$. Then, the trajectory sequence $\{x_m\}$ may not converge classically, but can still possess a rough I-statistical limit of order α'. If

$$x_m \xrightarrow{j'-\mathtt{I}\text{-}st(\alpha')} x^*,$$

then x^* represents an asymptotically stable neutrosophic equilibrium up to a tolerance level j'.

The parameter j' quantifies the maximum admissible uncertainty radius, while the order parameter α' controls the rate at which exceptional perturbations become negligible.

Therefore, rough I-statistical convergence of order α' provides a new tool for studying practical stability, robustness, and long-term behavior of uncertain dynamical systems modeled in neutrosophic normed spaces.

Theorem 3.6. *Let* $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ *be a neutrosophic normed space and consider the discrete-time dynamical system* $x_{m+1} = T(x_m) + e_m$*, where* $T : \mathcal{Y} \to \mathcal{Y}$ *is an operator and* e_m *is a disturbance sequence. Assume that:*

(i) $x^* \in \mathcal{Y}$ *is a fixed point of* T*, that is,* $T(x^*) = x^*$*;*
(ii) *the disturbance sequence is* I*-statistically bounded of order* α' *by* $j' > 0$*;*
(iii) *the unperturbed trajectory* $y_{m+1} = T(y_m)$ *is* I*-statistically convergent of order* α' *to* x^**.*

Then the perturbed trajectory x_m *is rough* I*-statistically convergent of order* α' *to* x^* *with roughness degree* j'*, that is,*

$$x_m \xrightarrow{,j'-\mathtt{I}\text{-}st(\alpha'),} x^*.$$

Consequently, x^* *is a rough* I*-statistically stable neutrosophic equilibrium of the system.*

Proof. Let

$$v_m = x_m - y_m,$$

so that

$$x_m = y_m + v_m \qquad (m \in \mathbb{N}).$$

By assumption (iii), for every $\varepsilon > 0$ and $\lambda^* \in (0,1)$,

$$\delta_I^*\Big(\big\{m \in \mathbb{N} : \Upsilon(y_m - x^*; \varepsilon) \le 1 - \lambda^* \text{ or } \Omega(y_m - x^*; \varepsilon) \ge \lambda^* \text{ or } \Gamma(y_m - x^*; \varepsilon) \ge \lambda^*\big\}\Big) = 0.$$

Since $\{e_m\}$ is $\mathtt{I}$-statistically bounded of order α' by j', it follows that

$$\delta_I^*\Big(\big\{m \in \mathbb{N} : \Upsilon(v_m; j') \le 1 - \lambda^* \text{ or } \Omega(v_m; j') \ge \lambda^* \text{ or } \Gamma(v_m; j') \ge \lambda^*\big\}\Big) = 0.$$

We define

$$A = \Big\{m \in \mathbb{N} : \Upsilon(y_m - x^*; \varepsilon) \le 1 - \lambda^* \text{ or } \Omega(y_m - x^*; \varepsilon) \ge \lambda^* \text{ or } \Gamma(y_m - x^*; \varepsilon) \ge \lambda^*\Big\},$$

and

$$B = \Big\{m \in \mathbb{N} : \Upsilon(v_m; j') \le 1 - \lambda^* \text{ or } \Omega(v_m; j') \ge \lambda^* \text{ or } \Gamma(v_m; j') \ge \lambda^*\Big\}.$$

Then,

$$\delta_I^*(A) = 0, \qquad \delta_I^*(B) = 0,$$

and therefore

$$\delta_I^*(A \cup B) = 0.$$

Let $m \notin A \cup B$. Then,

$$\Upsilon(y_m - x^*; \varepsilon) > 1 - \lambda^*, \qquad \Omega(y_m - x^*; \varepsilon) < \lambda^*, \qquad \Gamma(y_m - x^*; \varepsilon) < \lambda^*,$$

and

$$\Upsilon(v_m; j') > 1 - \lambda^*, \qquad \Omega(v_m; j') < \lambda^*, \qquad \Gamma(v_m; j') < \lambda^*.$$

Since

$$x_m - x^* = (y_m - x^*) + v_m,$$

the axioms of a neutrosophic norm imply

$$\begin{aligned}\Upsilon(x_m - x^*; j' + \varepsilon) &\ge \min\Big\{\Upsilon(y_m - x^*; \varepsilon), \Upsilon(v_m; j')\Big\} \\ &> 1 - \lambda^*,\end{aligned}$$

$$\begin{aligned}\Omega(x_m - x^*; j' + \varepsilon) &\le \max\Big\{\Omega(y_m - x^*; \varepsilon), \Omega(v_m; j')\Big\} \\ &< \lambda^*,\end{aligned}$$

and

$$\begin{aligned}\Gamma(x_m - x^*; j' + \varepsilon) &\le \max\Big\{\Gamma(y_m - x^*; \varepsilon), \Gamma(v_m; j')\Big\} \\ &< \lambda^*.\end{aligned}$$

Hence,

$$\Big\{m \in \mathbb{N} : \Upsilon(x_m - x^*; j' + \varepsilon) \le 1 - \lambda^* \text{ or } \Omega(x_m - x^*; j' + \varepsilon) \ge \lambda^* \text{ or } \Gamma(x_m - x^*; j' + \varepsilon) \ge \lambda^*\Big\} \subseteq A \cup B.$$

Therefore,

$$\begin{aligned}\delta_I^*\Big(\big\{m \in \mathbb{N} :\, &\Upsilon(x_m - x^*; j' + \varepsilon) \le 1 - \lambda^* \\ &\text{or } \Omega(x_m - x^*; j' + \varepsilon) \ge \lambda^* \\ &\text{or } \Gamma(x_m - x^*; j' + \varepsilon) \ge \lambda^*\big\}\Big) = 0.\end{aligned}$$

Since $\varepsilon > 0$ and $\lambda^* \in (0,1)$ are arbitrary, we obtain

$$x_m \xrightarrow{j'-\mathtt{I}\text{-}st(\alpha')} x^*.$$

Thus, x^* is a rough I-statistically stable equilibrium point of the neutrosophic dynamical system. □

3.3. **Application to Neutrosophic Sensor Networks and Data Fusion.** Modern intelligent systems frequently collect information from large-scale sensor networks operating in uncertain environments. Examples include environmental monitoring, smart cities, medical diagnosis, industrial automation, and autonomous vehicles. In such systems, sensor measurements are often contaminated by random noise, communication failures, and occasional abnormal observations.

Classical convergence methods are inadequate for analyzing such data because a small number of extreme measurements may destroy ordinary convergence. Furthermore, uncertainty in sensor readings is naturally represented through the three components of a neutrosophic norm: truth-membership, indeterminacy, and falsity.

Let y'_m denote the fused output of a sensor network at time m and suppose that ξ represents the actual physical quantity being monitored. Assume that the measurements satisfy

$$y'_m = \xi + \eta_m + \zeta_m,$$

where

(1) η_m is a small stochastic perturbation,
(2) ζ_m represents rare abnormal measurements caused by sensor malfunction, communication loss, or external disturbances.

If the set of abnormal observations

$$A = \{m : \zeta_m \neq 0\}$$

has I-density of order α' equal to zero, then the sequence $\{y'_m\}$ may fail to converge in the ordinary neutrosophic sense, but it can still be rough I-statistically convergent of order α'.

Consequently, there exists a roughness parameter $j' > 0$ such that

$$y'_m \xrightarrow{j'-\mathtt{I}\text{-}st(\alpha')} \xi.$$

This means that almost all sensor outputs remain within a neutrosophic tolerance radius j' of the true state ξ, while the exceptional measurements belong only to an I-small set.

Therefore, the rough I-statistical limit set

$$\mathtt{I}\text{-}st\text{-}LIM_{j'}^{\alpha'}(y)$$

provides a reliable estimate of the actual system state even in the presence of indeterminate and sporadic faulty observations.

Hence, rough I-statistical convergence of order α' offers a mathematically rigorous framework for robust data fusion, fault-tolerant sensor analysis, and uncertainty-aware decision making in neutrosophic environments.

Theorem 3.7. *Let $(\mathcal{Y}, \Upsilon, \Omega, \Gamma)$ be a neutrosophic normed space and let $\{y'_m\}$ be a sequence of fused sensor measurements given by*

$$y'_m = \xi + \eta_m + \zeta_m,$$

where $\xi \in \mathcal{Y}$ is the true state of the monitored system, $\{\eta_m\}$ is a measurement-noise sequence, and $\{\zeta_m\}$ is an abnormal-error sequence.

Assume that

(i) $\eta_m \xrightarrow{\mathtt{I}\text{-}st(\alpha')} 0$;
(ii) *the set $A = \{m \in \mathbb{N} : \zeta_m \neq 0\}$ has I-density of order α' equal to zero;*

(iii) *there exists* $j' > 0$ *such that* $\{\zeta_m\}$ *is* $\mathtt{I}$*-statistically bounded of order* α' *by* j'.

Then, the fused output sequence $\{y'_m\}$ *is rough* $\mathtt{I}$*-statistically convergent of order* α' *to* ξ*, namely*

$$y'_m \xrightarrow{j'-\mathtt{I}\text{-}st(\alpha')} \xi.$$

Consequently,

$$\xi \in \mathtt{I}\text{-}st\text{-}\mathrm{LIM}_{j'}^{\alpha'}(y),$$

and the sensor network provides a robust estimate of the true state despite sporadic abnormal observations.

Proof. Define

$$u_m = \xi + \eta_m, \qquad v_m = \zeta_m.$$

Then,

$$y'_m = u_m + v_m.$$

Since

$$\eta_m \xrightarrow{\mathtt{I}\text{-}st(\alpha')} 0,$$

it follows immediately that

$$u_m = \xi + \eta_m \xrightarrow{\mathtt{I}\text{-}st(\alpha')} \xi.$$

Indeed, for every $\varepsilon > 0$ and $\lambda^* \in (0,1)$,

$$\delta_I^*\Big(\{m : \Upsilon(\eta_m;\varepsilon) \le 1-\lambda^* \text{ or } \Omega(\eta_m;\varepsilon) \ge \lambda^* \text{ or } \Gamma(\eta_m;\varepsilon) \ge \lambda^*\}\Big) = 0,$$

which is equivalent to

$$\delta_I^*\Big(\{m : \Upsilon(u_m-\xi;\varepsilon) \le 1-\lambda^* \text{ or } \Omega(u_m-\xi;\varepsilon) \ge \lambda^* \text{ or } \Gamma(u_m-\xi;\varepsilon) \ge \lambda^*\}\Big) = 0.$$

Hence, $\{u_m\}$ is $\mathtt{I}$-statistically convergent of order α' to ξ.

Next, by assumption (iii), the abnormal-error sequence $v_m = \zeta_m$ is $\mathtt{I}$-statistically bounded of order α' by j'. Thus, for every $\lambda^* \in (0,1)$ there exists $\mathcal{H} > 0$ such that

$$\left\{ n \in \mathbb{N} : \frac{1}{n^{\alpha'}} \left| \left\{ m \le n : \Upsilon(v_m;\mathcal{H}) \le 1-\lambda^* \text{ or } \Omega(v_m;\mathcal{H}) \ge \lambda^* \text{ or } \Gamma(v_m;\mathcal{H}) \ge \lambda^* \right\} \right| \ge \delta^* \right\} \in \mathtt{I}.$$

Therefore, $\{v_m\}$ is $\mathtt{I}$-statistically bounded of order α'.

Since

$$y'_m = u_m + v_m,$$

where $\{u_m\}$ is $\mathtt{I}$-statistically convergent of order α' to ξ and $\{v_m\}$ is $\mathtt{I}$-statistically bounded of order α' by j', By the Theorem 3.5 yields

$$y'_m \xrightarrow{j'-\mathtt{I}\text{-}st(\alpha')} \xi.$$

Equivalently, for every $\varepsilon > 0$ and $\lambda^* \in (0,1)$,

$$\delta_I^*\Big(\{m : \Upsilon(y'_m-\xi;j'+\varepsilon) \le 1-\lambda^* \text{ or } \Omega(y'_m-\xi;j'+\varepsilon) \ge \lambda^* \text{ or } \Gamma(y'_m-\xi;j'+\varepsilon) \ge \lambda^*\}\Big) = 0.$$

Hence,

$$\xi \in \mathtt{I}\text{-}st\text{-}\mathrm{LIM}_{j'}^{\alpha'}(y).$$

Therefore, the fused sensor output remains asymptotically inside a neutrosophic tolerance region of radius j' around the true state ξ, except on an $\mathtt{I}$-small set of indices of order α'. This proves the robustness of the data-fusion process against sporadic faulty measurements. □

Example 3.4. *Consider the neutrosophic normed space* $(\mathbb{R}, \Upsilon, \Omega, \Gamma)$, *where*

$$\Upsilon(x;t) = \frac{t}{t+|x|}, \qquad \Omega(x;t) = \frac{|x|}{2(t+|x|)}, \qquad \Gamma(x;t) = \frac{|x|}{2(t+|x|)}$$

for all $x \in \mathbb{R}$ *and* $t > 0$.

Let $\mathtt{I} = \mathtt{I}_f$ *be the ideal of all finite subsets of* $\mathbb{N}$ *and let the true state of the monitored system be* $\xi = 10$.

Define the measurement-noise sequence

$$\eta_m = \frac{(-1)^m}{m},$$

and the abnormal-error sequence

$$\zeta_m = \begin{cases} \sqrt{m}, & \text{if } m = k^2 \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

The fused sensor output is

$$y'_m = 10 + \frac{(-1)^m}{m} + \zeta_m.$$

We verify the assumptions of the theorem 3.7.

Step 1. *Since* $\lim_{m\to\infty} \frac{(-1)^m}{m} = 0$, *it follows that* $\eta_m \xrightarrow{\mathtt{I}_f\text{-}st(1)} 0$.

Step 2. *The set of abnormal observations is*

$$A = \{1, 4, 9, 16, \ldots\}.$$

Since

$$|A \cap \{1, \ldots, n\}| = \lfloor \sqrt{n} \rfloor,$$

we obtain

$$\lim_{n\to\infty} \frac{\lfloor \sqrt{n} \rfloor}{n} = 0.$$

Hence, A *has* $\mathtt{I}_f$*-density of order* 1 *equal to zero.*

Step 3. *Choose* $j' = 1$. *For every non-square index* m,

$$\zeta_m = 0.$$

Therefore,

$$\Upsilon(\zeta_m; 1) = 1, \qquad \Omega(\zeta_m; 1) = 0, \qquad \Gamma(\zeta_m; 1) = 0.$$

The exceptional indices are precisely the square numbers. Consequently, for every $\lambda^* \in (0, 1)$,

$$\Big|\{m \le n : \Upsilon(\zeta_m; 1) \le 1 - \lambda^* \text{ or } \Omega(\zeta_m; 1) \ge \lambda^* \text{ or } \Gamma(\zeta_m; 1) \ge \lambda^*\}\Big| \le \sqrt{n}.$$

Hence,

$$\frac{\sqrt{n}}{n} \longrightarrow 0,$$

which proves that $\{\zeta_m\}$ *is* $\mathtt{I}_f$*-statistically bounded of order* 1 *by* $j' = 1$.

All assumptions of the theorem are satisfied. Therefore,

$$y'_m \xrightarrow{1-\mathtt{I}_f\text{-}st(1)} 10.$$

Thus,

$$10 \in \mathtt{I}_f\text{-}st\text{-}\mathrm{LIM}_1^1(y).$$

Notice that the sequence $\{y'_m\}$ *is not ordinarily convergent because*

$$y'_{k^2} = 10 + \frac{(-1)^{k^2}}{k^2} + k \longrightarrow \infty.$$

Nevertheless, the abnormal measurements occur only on a set of natural density zero. Hence the sensor network still recovers the true state $\xi = 10$ in the sense of rough I*-statistical convergence. This illustrates the robustness of neutrosophic data fusion against sporadic large sensor failures.*

4. Conclusion

We investigated rough I-statistical convergence of order α' in the setting of neutrosophic normed spaces. We established the existence of a minimal roughness parameter for which the corresponding I-statistical limit set is nonempty and showed that this convergence is preserved under small perturbations of sequences. It was further proved that whenever the intersection of all rough I-statistical limit sets is nonempty, it consists of a unique point. Additionally, we characterized rough I-statistical convergence of order α' in terms of subsequences indexed by filter sets and by decomposing sequences into I-statistically convergent and I-statistically bounded parts.

The proposed framework extends classical concepts of rough, statistical, and ideal convergence to the neutrosophic setting. Furthermore, applications to neutrosophic dynamical systems, perturbed Picard iterations, and sensor-network data fusion demonstrate the usefulness of the theory in studying stability, robustness, and uncertainty. Future work may focus on extensions to operator theory, fixed-point theory, sequence spaces, and other neutrosophic structures.

References

[1] S. Aytar, Rough statistical convergence, Numer. Funct. Anal. Optim., 29(3–4) (2008), 291–303.
[2] P. Kostyrko, T. Šalát and W. Wilczyński, I-convergence, Real Anal. Exchange, 26(2) (2000), 669–685.
[3] P. Das, E. Şavas and S. K. Ghosal, On generalizations of certain summability methods using ideals, Appl. Math. Lett., 24(9) (2011), 1509–1514.
[4] H. X. Phu, Rough convergence in normed linear spaces, Numer. Funct. Anal. Optim., 22(1–2) (2001), 199–222.
[5] H. X. Phu, Rough convergence in infinite dimensional normed spaces, Numer. Funct. Anal. Optim., 24 (2003), 285–301.
[6] M. Maity, A note on rough statistical convergence of order α, J. Pure Math., 31 (2014), 37–46.
[7] S. K. Pal, Ch. Debraj and S. Dutta, Rough ideal convergence, Hacet. J. Math. Stat., 42(6) (2013), 633–640.
[8] S. Bulut and A. Or, I-statistical rough convergence of order α, J. New Theory, 38 (2022), 34–41.
[9] M. Ahmad, E. Savas and M. Mursaleen. On deferred I-statistical rough convergence of difference sequences in neutrosophic normed spaces, Neutrosophic Sets and Systems, 96 (2026), 224-262.
[10] M. Ahmad, M. Mursaleen and E. Savaş, Rough I-γ_τ-statistical convergence of order α in neutrosophic normed spaces, Journal of Intelligent & Fuzzy Systems, 50(2026).
[11] F. Smarandache, Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic probability, Sitech, Craiova, 2013.
[12] M. Kirişci and N. Şimşek, Neutrosophic normed space and statistical convergence, J. Anal., 28(4) (2020), 1059–1073.
[13] V. A. Khan, H. Fatima, M. D. Khan and A. Ahmad, Spaces of neutrosophic λ-statistical convergence sequences and their properties, J. Math. Comput. Sci., 23(1) (2021), 1–9.
[14] V. A. Khan, M. Et and M. Faisal, Some new neutrosophic normed sequence spaces defined by Jordan totient operator, Filomat, 37(26) (2023), 8953–8968.
[15] R. Antal, M. Chawla and V. Kumar, Rough statistical convergence in intuitionistic fuzzy normed spaces, Filomat, 35(13) (2021), 4405–4416.
[16] R. Antal, M. Chawla and V. Kumar, Rough statistical convergence in probabilistic normed spaces, Thai J. Math., 20(4) (2022), 1707–1719.
[17] M. Arslan and E. Dündar, Rough convergence in 2-normed spaces, Bull. Math. Anal. Appl., 10(3) (2018), 1–9.
[18] M. Arslan and E. Dündar, Rough statistical convergence in 2-normed spaces, Honam Math. J., 43(3) (2021), 417–431.
[19] M. Arslan and E. Dündar, Rough statistical cluster points in 2-normed spaces, Thai J. Math., 20(3) (2022), 1419–1429.

[20] E. Dündar, On Rough I_2-Convergence of Double Sequences, Numer. Funct. Anal. Optim., 37(4) (2016), 480–491.
[21] E. Dündar and C. Çakan, Rough I-convergence, Demonstr. Math., 47(3) (2014), 638–651.
[22] E. Dündar and U. Ulusu, On Rough I-Convergence and I-Cauchy Sequence for Functions Defined on Amenable Semigroups, Univ. J. Math. Appl., 6(2) (2023), 86–90.
[23] E. Dündar and U. Ulusu, Rough Statistical Convergent Functions Defined on Amenable Semigroups, Proc. Natl. Acad. Sci. India Sect. A, 94(3) (2024), 317–323.
[24] F. Lael and K. Nourouzi, Some results on the intuitionistic fuzzy normed spaces, Chaos Solitons Fractals, 37(3) (2008), 931–939.
[25] S. Karakuş, K. Demirci and O. Duman, Statistical convergence on intuitionistic fuzzy normed spaces, Chaos Solitons Fractals, 35(4) (2008), 763–769.
[26] J. H. Park, Intuitionistic fuzzy metric spaces, Chaos Solitons Fractals, 22(5) (2004), 1039–1046.
[27] R. Saadati and J. H. Park, On the intuitionistic fuzzy topological spaces, Chaos Solitons Fractals, 27(2) (2006), 331–344.
[28] M. Mursaleen and Q. D. Lohani, Intuitionistic fuzzy 2-normed space and some related concepts, Chaos Solitons Fractals, 42 (2009), 224–234.
[29] M. Mursaleen and S. A. Mohiuddine, On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space, J. Comput. Appl. Math., 233 (2009), 142–149.
[30] V. Kumar and M. Mursaleen, On (λ, μ)-statistical convergence of double sequences on intuitionistic fuzzy normed spaces, Filomat, 25 (2011), 109–120.
[31] V. A. Khan, M. Faisal and M. Kamran, Some topological properties of intuitionistic fuzzy quasi normed space, Filomat, 38(25) (2024), 8907–8916.
[32] V. A. Khan and M. Faisal, A study on $(\lambda-\mu)$ Zweier sequences and their behaviour in neutrosophic normed spaces, Neutrosophic Sets Syst., 73(1) (2024), Article 47.
[33] V. A. Khan and M. Faisal, Tauberian theorem for Cesàro summability in neutrosophic normed spaces, Filomat, 37(11) (2023), 3411–3426.
[34] M. Ahmad and M. Mursaleen, On deferred statistical summability in neutrosophic n-normed linear space, Filomat, 39(18) (2025), 6123–6148.
[35] M. Ahmad and M. Mursaleen, Completeness and Cartesian Product in Neutrosophic Rectangular n-Normed Spaces, Dera Natung Government College Research Journal, 10(1) (2025), 22–40.